Exercise 04

Name:	Matriculation number:
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The exercise is due on Wednesday, May 16, 8 am.

4.1 Probability theory of alphabet pasta

Table 1 shows the relative frequency (in percent) of letters in the English language. In a random experiment, N letters are drawn with probabilities given by this distribution. (Imagine sitting in front of a enormous bag of alphabet noodles and picking out noodles, one at a time.) The outcome of the experiment is a sequence L of letters, e.g.

$$L = \{\mathrm{p, r, o, b, a, b, i, l, i, t, y}\}\,.$$

The composition C of a sequence denotes the how often a given letter occurs in the sequence

$$C = \mathrm{p_1 r_1 o_1 b_2 a_1 i_2 l_1 t_1 y_1}\,.$$

Swapping letters changes the sequence, but not the composition. (e.g. $L' = \{\mathrm{p, r, o, b, i, b, a, l, i, t, y}\}$)

(a) What is the most likely sequence of length N=5?

(b) For an arbitrary sequence of letters, give general expressions for the number of possible sequences with the same composition, the probability of the sequence $\mathbb{P}(L)$, and the probability of the composition $\mathbb{P}(C)$.

(c) Approximate $\mathbb{P}(C)$ by taking the logarithm, and applying Stirling's formula

$$\begin{aligned} \ln(N!) &\approx \ln\left(\frac{N^N}{e^N}\sqrt{2\pi N}\right) = N\ln N - N + \frac{1}{2}\ln(2\pi N) \\ &\approx N\ln N - N \end{aligned} \tag{1}$$

The resulting quantity is called the entropy $S(C)$.

(d) For the words

- statistical
- thermodynamics
- statistical thermodynamics

Calculate the number of possible sequences with the same composition, the probability of the sequence $\mathbb{P}(L)$, the probability of the composition $\mathbb{P}(C)$, the logarithm of $\mathbb{P}(C)$, and the entropy $S(C)$.

(e) Comment on the values for the entropy $S(C)$ and for $\mathbb{P}(C)$

Hint: disregarde the space in "statistical thermodynamics".

Letter	Frequency (%)	Letter	Frequency (%)	Letter	Frequency (%)	Letter	Frequency (%)
a	8.167	h	6.094	o	7.507	v	0.978
b	1.492	i	6.966	p	1.929	w	2.360
c	2.782	j	0.153	q	0.095	x	0.150
d	4.253	k	0.772	r	5.987	y	1.974
e	12.702	l	4.025	s	6.327	z	0.074
f	2.228	m	2.406	t	9.056		
g	2.015	n	6.749	u	2.758		

Table 1: Relative frequency of letters in the English language.

4.2 Temperature dependency of the partition function

In general the partition function can be expressed as:

$$Z(T) = \sum_{i=0}^{\infty} g_i e^{-\frac{E_i}{k_b T}} \tag{2}$$

where g_i denotes the degeneracy factor of energy level i.

(a) What do you observe for $T \to 0$? Explain your results and calculate the partition function.

(b) Repeat (a) for $T \to \infty$.

(c) For which case does the partition function for $T \to \infty$ have a finite limit?

4.3 Calculation of energy levels

Consider a system at 25 °C with 4 accessible energy levels. Now assume that the energy levels are populated with

$$p_0 = 0.8, p_1 = 0.1, p_2 = 0.05, p_3 = 0.05.$$

(a) Calculate the energy of the first 4 energy levels as well as the energy difference between them if $E_0 = 10^{-20}$ J.

(b) Draw an energy scheme. What do you observe?

(c) Consider E_2 and E_3 as a single energy level E_2' with $g_2 = 2$. At which temperature are the energy levels E_2' and E_1 equally populated? At which temperature E_2' and E_0?

(d) Is there a finite temperature at which E_0 and E_1 are populated equally? Justify your answer.