Notes on classical electromagnetism

1 Maxwell's equations

The Maxwell's equations are a set of partial differential equations that summarize the classical electromagnetism. These equations give us information about the properties of the electric and magnetic field and how they are related.

1. The first equation tells us that the electric field generated by charged particles "diverges", i.e. that the field lines are "open", starting or arriving to the source according to the charge sign.
$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \tag{1}$$

2. On the other hand, the second equation tells us that the magnetic field does not diverge and it is solenoidal. This means that the field lines are "close", they start from the North and arrive to the South pole. Another important consequence of this equation is that there not exist magnetic monopoles.
$$\nabla \cdot \mathbf{B} = 0 \tag{2}$$

3. The third equation tells us how the variation with time of the magnetic field induces an electric field.
$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{3}$$

4. The forth equation tells us that both a stationary current and a variable electric field can induce a magnetic field.
$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \tag{4}$$

2 Vector potential and scalar potential

Exploiting some mathematical proprieties of the divergence and of the curl, we can define two new objects, starting from the Maxwell's equation, useful for a more complete description of the electromagnetism.
Using $\nabla \cdot \nabla \times \mathbf{v} = 0 \;\; \forall$ vector $\mathbf{v}$ (i.e. *the divergence of a curl is zero*), from the second Maxwell's equation (2), we can define a new vector called *vector potential*:

$$\mathbf{B} = \nabla \times \mathbf{A} \tag{5}$$

Plugging the equation (5) in the third Maxwell's equation (3), we get:

$$\nabla \times \mathbf{E} = -\frac{\partial \nabla \times \mathbf{A}}{\partial t}$$
$$\nabla \times \left(\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} \right) = 0 \tag{6}$$

Another mathematical property of the curl is $\nabla \times \nabla s = 0 \; \forall$ scalar s (i.e. *the curl of a scalar is zero*).
Then the equation (6) tells us that the term between round brackets is the gradient of some scalar function:

$$\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} = -\nabla V \tag{7}$$

where V is called *scalar potential.*
Note that this potential is the same found in the case stationary (i.e. in electrostatics), where the third Maxwell's equation (3) becomes $\nabla \times \mathbf{E} = 0$ and $\mathbf{E} = -\nabla V$.

The choice of the vector potential and of the scalar potential is not unique. Expoiting again $\nabla \times \nabla s = 0 \; \forall$ scalar s, if $\mathbf{A}$ is a satisfactory potential vector such that $\mathbf{B} = \nabla \times \mathbf{A}$, then for any scalar ϕ,

$$\mathbf{A}' = \mathbf{A} + \nabla\phi$$

will be an equally satisfactory potential vector for the same vector field $\mathbf{B}$.
In a similar way, exploiting $\nabla C = 0$ for any constant C, from the potential V, we can define a new scalar potential:

$$V' = V + C$$

3 Vector potential of constant magnetic field

We can exploit the non-uniqueness of the vector potential, to make a mathematically convenient choice according to our problem.
First of all, we can restrict our vector potential choosing what the divergence $\mathbf{A}$ must to be. In fact, even though $\mathbf{A}$ and $\mathbf{A}'$ have the same curl, they do not have the same divergence. In the case of a constant magnetic field, it is convenient to choose $\mathbf{A}$ such that,

$$\nabla \cdot A = 0$$

This choice is called *Coulomb gauge.*
Let's now consider the case of a constant magnetic field $\mathbf{B} = (0, 0, B_0)$. From the equation (5) $\mathbf{B} = \nabla \times \mathbf{A}$, we have:

$$B_x = \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} = 0$$

$$B_y = \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} = 0$$

$$B_z = \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} = B_0$$

These three equations have several solutions. One possibility is:

$$A_x = 0, \quad A_y = xB_0, \quad A_z = 0$$

A second possible solution is:

$$A_x = -yB_0, \quad A_y = 0, \quad A_z = 0$$

Finally we can construct a third solution as linear combination of the first two:

$$A_x = -\frac{1}{2}yB_0, \quad A_y = \frac{1}{2}xB_0, \quad A_z = 0$$

that can be written as

$$\mathbf{A} = -\frac{1}{2}(\mathbf{r} \times \mathbf{B}) \tag{8}$$

where $\mathbf{r} = (x, y, z)$ is the distance from the z-axis. This particular solution tells us that the vector potential (for a uniform magnetic field along the z-direction) rotates about the z-axis and its magnitude $\frac{Br^2}{2}$ is proportional to the distance from the z-axis.

4 The electromagnetic Hamiltonian

Finally, we will derive the Hamiltonian equation for a particle with charge q in an electromagnetic field using the potential vector.
First of all, we need to introduce another important law of the electromagnetism, called *Lorentz force law*:

$$\mathbf{F} = q\left(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B}\right) \tag{9}$$

where c is the speed of light (from now we will use $c = 1$) and $\mathbf{v}$ is the velocity of the particle.
This law tells us that the particle is subject to a force proportional to the electric field and that travelling in a magnetic field, it deviates its trajectory perpendicularly to the instantaneous velocity and the magnetic field.
We use the equations (5) and (7), to rewrite the Lorents force law as a function of the vector and scalar potential:

$$\mathbf{F} = q\left(-\nabla V - \frac{\partial \mathbf{A}}{\partial t} + \mathbf{v} \times (\nabla \times \mathbf{A})\right) \tag{10}$$

We now use the vector identity

$$\mathbf{g} \times (\nabla \times \mathbf{h}) = \nabla(\mathbf{g} \cdot \mathbf{h}) - (\mathbf{g} \cdot \nabla)\mathbf{h} - (\mathbf{h} \cdot \nabla)\mathbf{g} - \mathbf{h} \times (\nabla \times \mathbf{g})$$

Because $\mathbf{v}$ is not an explicit function of position, the third and the forth term of the last identity are zeros, so we have:

$$\mathbf{v} \times (\nabla \times \mathbf{A}) = \nabla(\mathbf{v} \cdot \mathbf{A}) - (\mathbf{v} \cdot \nabla)\mathbf{A}$$

We now write the total derivative of the vector potential

$$\frac{d\mathbf{A}}{dt} = \frac{\partial \mathbf{A}}{\partial t} + \frac{\partial \mathbf{A}}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial \mathbf{A}}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial \mathbf{A}}{\partial z}\frac{\partial z}{\partial t} = \frac{\partial \mathbf{A}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{A}$$

and we can rewrite the Lorentz force as:

$$\mathbf{F} = q\left(-\nabla V - \frac{d\mathbf{A}}{dt} + \nabla(\mathbf{v} \cdot \mathbf{A})\right) \tag{11}$$

When a force depend on the velocity (like the Lorentz force), we can write it as a function of some generalized potential U:

$$\mathbf{F} = -\nabla U + \frac{d}{dt}\left(\nabla_{\mathbf{v}} U\right) \tag{12}$$

Note that if the force does not depend on the velocity, we get the usual relation $\mathbf{F} = -\nabla U$.
To write the Lorentz force (11) in the form (12), we write the derivative of the vector potential as:

$$\frac{d\mathbf{A}}{dt} = \frac{d}{dt}\left(\nabla_{\mathbf{v}}(\mathbf{v} \cdot \mathbf{A})\right) \tag{13}$$

then we exploit $\nabla_{\mathbf{v}}(qV) = 0$, to insert the scalar potential inside the round brackets of(13):

$$\frac{d\mathbf{A}}{dt} = \frac{d}{dt}\left(\nabla_{\mathbf{v}}(\mathbf{v} \cdot \mathbf{A} - qV)\right)$$

We can now rewrite the equation (11) as

$$\mathbf{F} = q\left(-\nabla V - \frac{d}{dt}\left(\nabla_{\mathbf{v}}(\mathbf{v} \cdot \mathbf{A} - qV)\right) + \nabla(\mathbf{v} \cdot \mathbf{A})\right)$$

$$\mathbf{F} = -\nabla(qV - q(\mathbf{v} \cdot \mathbf{A})) + \frac{d}{dt}\left(\nabla_{\mathbf{v}}(qV - q(\mathbf{v} \cdot \mathbf{A}))\right)$$

then our generalized potential is

$$U = qV - q(\mathbf{v} \cdot \mathbf{A})$$

Finally we can write the electromagnetic Lagrangian

$$L = T - U = \frac{1}{2}m\mathbf{v} \cdot \mathbf{v} - qV + q(\mathbf{v} \cdot \mathbf{A}),$$

the canonical momentum

$$\mathbf{p} = \frac{\partial L}{\partial \mathbf{v}} = m\mathbf{v} + q\mathbf{A}$$

and the electromagnetic Hamiltonian:

$$H = \mathbf{p} \cdot \mathbf{v} - L = \frac{1}{2m}\left(\mathbf{p} - q\mathbf{A}\right)^2 + qV$$