Exercise 08

1 Angular momentum

1. Show that

$$[\hat{L}^2, \hat{L}_x] = [\hat{L}^2, \hat{L}_y] = [\hat{L}^2, \hat{L}_z] \quad = \quad 0 \tag{1}$$

where

$$\hat{L}^2 \quad = \quad \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2 \,. \tag{2}$$

2 Angular momentum: ladder operators

In this exercise, you will practice the use of angular momentum ladder operators, which are given as:

$$\begin{aligned} \hat{L}_+ &= \hat{L}_x + i\hat{L}_y \\ \hat{L}_- &= \hat{L}_x - i\hat{L}_y \end{aligned} \tag{3}$$

First we show some general relations.

1. Show that

$$\left[\hat{L}_z, \hat{L}_\pm\right] \quad = \quad \pm\hbar\hat{L}_\pm \tag{4}$$

2. Show that

$$\hat{L}_\pm\hat{L}_\mp \quad = \quad \hat{L}^2 - \hat{L}_z^2 \pm \hbar\hat{L}_z \tag{5}$$

Next we want the show that $\hat{L}_+$ raises the eigenfunction of $\hat{L}_z$ by one, and that $\hat{L}_-$ lowers the eigenfunction of of $\hat{L}_z$ by one.

3. Given an eigenfunction ψ_{lm} of $\hat{L}_z$, with eigenvalue $\hbar m$:

$$\hat{L}_z\psi_{lm} \quad = \quad \hbar m\psi_{lm} \tag{6}$$

Show that:

$$\hat{L}_z\hat{L}_\pm\psi_{lm} \quad = \quad \hbar(m \pm 1)\hat{L}_\pm\psi_{lm} \tag{7}$$

Eq. 6 implies that if ψ_{lm} is an eigenfunction of $\hat{L}_z$ with eigenvalue $\hbar m$, then $L_+\psi_{lm}$ is an eigenfunction with eigenvalue $\hbar(m+1)$. Analogously, if ψ_{lm} is an eigenfunction of $\hat{L}_z$ with eigenvalue $\hbar m$, then $L_-\psi_{lm}$ is an eigenfunction with eigenvalue $\hbar(m-1)$. One can construct more eigenfunctions be applying the ladder operators several times.

4. Show that

$$\hat{L}_z\hat{L}_\pm^k\psi_{lm} \quad = \quad \hbar(m \pm k)\hat{L}_\pm^k\psi_{lm} \,. \tag{8}$$

Because $[\hat{L}^2, \hat{L}_z] = 0$, the thus constructed functions ψ_{ml} are also eigenfunctions to $\hat{L}^2$

$$\hat{L}^2\psi_{lm} = c^2\psi_{lm} \,. \tag{9}$$

Now we show that all functions which have been constructed by applying $\hat{L}_\pm$ to an initial function ψ_{lm} have the same eigenvalue with respect to $\hat{L}^2$, i.e.

$$\hat{L}^2\hat{L}_\pm^k\psi_{lm} = c^2\hat{L}_\pm^k\psi_{lm} \qquad k = 0, 1, 2, ... \tag{10}$$

5. Show that

$$[\hat{L}^2, \hat{L}_\pm] = 0 \tag{11}$$

and that

$$[\hat{L}^2, \hat{L}_\pm^2] = 0\,. \tag{12}$$

By induction, it follows from eqs. 11 and 12 that

$$[\hat{L}^2, \hat{L}_\pm^k] = 0\,. \tag{13}$$

6. Prove eq. 10 using eq. 13.

Write a report by hand and deliver it at the office 35.17 before next Wednesday at noon.