Exercise 03

1 Hermitian operators (15 points)

1. Consider an operator $\hat{A}$ with degenerate eigenvectors $|\phi_i\rangle$ (i.e. with eigenvectors with the same eigenvalues). Prove that any linear combination of these eigenvectors are also eigenvectors of the operator $\hat{A}$ with the same eigenvalue.
2. Prove that Hermitian operators have real eigenvalues.
3. Prove that Hermitian operators have orthogonal eigenvectors.

2 Dirac notation (10 points)

1. Consider an orthonormal basis:

$$\{|1\rangle, |2\rangle, |3\rangle\} = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\} \tag{1}$$

and the vectors:

$$|a_1\rangle = 2|1\rangle + 1|2\rangle + 1|3\rangle$$
$$|a_2\rangle = -1|1\rangle + 0|2\rangle + 1|3\rangle$$
$$|a_3\rangle = 0|1\rangle + 2|2\rangle + 3|3\rangle$$

Show that the vectors $|a_1\rangle, |a_2\rangle$ and $|a_3\rangle$ are linearly independent.
Consider the vectors:

$$|b_1\rangle = 2i|1\rangle + 0|2\rangle - 1|3\rangle$$
$$|b_2\rangle = 0|1\rangle - i|2\rangle + 3|3\rangle$$

Compute: $\langle b_1|, \langle b_2|, \langle b_1|b_2\rangle, |b_1\rangle\langle b_2|$

2. Given the operator $\hat{A}$, that acts on the vector basis as follows:

$$\hat{A}|1\rangle = 3|1\rangle + i|2\rangle$$
$$\hat{A}|2\rangle = -i|1\rangle + 1|3\rangle + 2|3\rangle$$
$$\hat{A}|3\rangle = 1|2\rangle - 2i|2\rangle$$

find the matrix representation of $\hat{A}$. Is the operator Hermitian?

3 Commutators (10 points)

1. Consider the operators momentum $\hat{p}$ and position $\hat{x}$, prove that $[\hat{p}, \hat{x}] = -i\hbar$
2. The angular momentum is defined as $\mathbf{L} = \mathbf{r} \times \mathbf{p}$, with $\mathbf{r} = (\hat{x}, \hat{y}, \hat{z})$ and $\mathbf{p} = (\hat{p}_x, \hat{p}_y, \hat{p}_z)$. Using the cross product, shows that:

$$\hat{L}_x = \hat{y}\hat{p}_z - \hat{z}\hat{p}_y$$
$$\hat{L}_y = \hat{z}\hat{p}_x - \hat{x}\hat{p}_z$$
$$\hat{L}_z = \hat{x}\hat{p}_y - \hat{y}\hat{p}_x$$

Finally prove that:

$$\left[\hat{L}_x, \hat{L}_y\right] = i\hbar\hat{L}_z$$
$$\left[\hat{L}_y, \hat{L}_z\right] = i\hbar\hat{L}_x$$
$$\left[\hat{L}_z, \hat{L}_x\right] = i\hbar\hat{L}_y$$

4 Particle in a box (10 points)

1. Show that the wavefunctions of a particle in a box are orthonormal. Use the wavefunction:

$$\psi(x) = B \sin\left(\frac{n\pi}{L}x\right) \tag{2}$$

   Use this result to normalize the wavefunction and to find the value of B.

2. Show that the average momentum of a particle in a box is zero:

$$\langle \hat{p}_x \rangle = \int_0^L \psi(x)^* \hat{p}_x \psi(x) dx = 0 \tag{3}$$

For these exercises you do not have to use MATLAB, so you can write a report by hand and deliver it at the office 35.17 or write an electronic document and send it to luca.donati@fu-berlin.de before next Wednesday at noon.