Exercise 04

NAME:	MATRICULATION NUMBER:
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RESULTS:

4.1	OF 6 P
4.2	OF 12 P
4.3	OF 8 P
TOTAL	OF 26 P

GENERAL INSTRUCTIONS

- SUBMIT YOUR SOLUTION TO OLIVER LEMKE (R. 35.17) BEFORE **THURSDAY 26. MAY AT 8.15 AM**.
- FILL OUT THIS COVER SHEET AND SUBMIT IT ALONG WITH YOUR SOLUTION.
- SHOW HOW YOU ARRIVED AT YOUR ANSWER.

4.1 Approximation of the harmonic oscillator (6 P)

To approximate a harmonical potential from an arbitary function $V(x)$ a Taylor expansion around the stationary coordinate x_0 can be applied:

$$V(x - x_0) = \sum_{n=0}^{\infty} \frac{V^{(n)}(x_0)}{n!}(x - x_0)^n \tag{1}$$

This expansion is truncated after the third term.
Let us assume you have a potential of the form:

$$V(x) = \lambda(x^2 - a^2)^2$$

(a) Plot the potential

(b) Approximate $V(x)$ as a Taylor expansion around a

(c) Calculate the extrema of $V(x)$ as well as the extrema of the Taylor expansion of $V(x)$

4.2 Lagrange multiplier (12 P)

To calculate the extrema of a function $f(x, y)$ with constraints $g(x, y)$ the Lagrange formalism can be applied:

$$\mathcal{L}(x, y; \lambda) = f(x, y) + \lambda(c - g(x, y)) \tag{2}$$

where $c = g(x, y)$ and λ denotes the Lagrange multiplier. To solve the equation the partial derivative with respect to all parameters and variables is calculated:

$$\mathcal{L}_x = \frac{\partial \mathcal{L}}{\partial x} = f_x - \lambda g_x = 0 \tag{3}$$

$$\mathcal{L}_y = \frac{\partial \mathcal{L}}{\partial y} = f_y - \lambda g_y = 0 \tag{4}$$

$$\mathcal{L}_\lambda = \frac{\partial \mathcal{L}}{\partial \lambda} = c - g(x, y) = 0 \tag{5}$$

This results in a system of linear equations which can be solved.

(a) Calculate the extremum of the function

$$f(x,y) = x^2 + 2y^2 \text{ with } g(x,y) = x + y = 3$$

(b) To characterize the extremum further the bordered hessian can be analyzed

$$\mathbf{H}(f,g) = \begin{bmatrix} 0 & \frac{\partial g}{\partial x_1} & \frac{\partial g}{\partial x_2} & \cdots & \frac{\partial g}{\partial x_n} \\ \frac{\partial g}{\partial x_1} & \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1\,\partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1\,\partial x_n} \\ \frac{\partial g}{\partial x_2} & \frac{\partial^2 f}{\partial x_2\,\partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \cdots & \frac{\partial^2 f}{\partial x_2\,\partial x_n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial g}{\partial x_n} & \frac{\partial^2 f}{\partial x_n\,\partial x_1} & \frac{\partial^2 f}{\partial x_n\,\partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_n^2} \end{bmatrix} = \begin{bmatrix} 0 & \nabla g \\ \nabla g^\top & \mathbf{H} f \end{bmatrix} \tag{6}$$

which simplifies for 2 variables to

$$\mathbf{H}(f,g) = \begin{bmatrix} 0 & g_x & g_y \\ g_x & \mathcal{L}_x x & \mathcal{L}_x y \\ g_y & \mathcal{L}_y x & \mathcal{L}_y y \end{bmatrix} \tag{7}$$

Calculate the determinant of this matrix. If the determinant is smaller than 0 a minimum, for a determinant greater than 0 a maximum, is present. For $det|\mathbf{H}(f,g)| = 0$ no clear statement can be made.

(c) for the presence of more than 1 constraint the Lagrange formalism is given as:

$$\mathcal{L}(x,y;\lambda) = f(x,y) + \sum_{i=1}^{k} \lambda(c_i - g_i(x,y)) \tag{8}$$

where k is the number of constraints. Solve the Lagrange formalism for

$f(x,y,z) = (x-1)^2 + (y-2)^2 + 2z^2$ with $g_1(x,y,z) = x + 2y = 2$ and $g_2(x,y,z) = y - z = 3$

and calculate the extremum.