Exercise 01

Numerical Approximations

Deadline: Please, hand in your solutions by **Thursday, 15 May, 2.15 p.m.**

1.1 Series limit (6P)

In this exercise, you can practice the use of for-loops and do-while-loops. Consider the convergent series

$$\sum_{i=0}^{\infty} \frac{1}{2^i} = \frac{1}{1} + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots \,. \tag{1}$$

(a) Show that the limit of the series is 2. (Hint: Double each term in the series.) (1 P)

(b) Write a program which numerically estimates the limit of the series by evaluating the first N terms. Plot the estimate as a function of N up to $N = 20$. (2 P)

(c) Write a program which evaluates the series until the increment is less than a pre-defined threshold. How many terms are needed until the increment is less than 10^{-2}, 10^{-4}, 10^{-6}, 10^{-8}? (2 P)

(d) Often one does not know in advance how long it will take until a certain threshold is met. To prevent a program from running too long, one stops the calculation if either the threshold or a maximum number of iterations is reached. Modify the program from (c) such that it stops if one of the following conditions is met

- the increment is less than a predefined threshold,
- a maximum number of terms have been evaluated. (1 P)

1.2 Random number generator (4P)

(a) Generate three sequences of random real numbers drawn from the interval $[0, \pi]$. The sequences should have 100, 10.000, and 1.000.000 elements. Plot histograms of these sequences with a bin width of $\pi/20$. Is the distribution of the random numbers uniform? (2 P)

(b) From a sequence of N random numbers in the intervall $[0, \pi]$, one can estimate the value of the fraction $1/\pi$ as

$$\frac{1}{\pi} \approx \frac{n}{N}$$

where n is the number of elements in the sequence which are ≤ 1. Write a program which estimates $1/\pi$ in this way, and plot the estimate as function of N. Don't forget to add the Matlab-result of $1/\pi$ as a line to the plot. (2 P)

1.3 Numerical Integration of Functions: Trapezoidal Rule (8P)

Consider the function

$$f(x) \quad = \quad (x-2)^3 - 10 \cdot (x-3)^2 + 100 \,. \tag{2}$$

In this exercise, you will implement an program which evaluates the integral over $f(x)$ numerically using the trapezoid rule. The trapezoidal rule states that the integral of a function $f(x)$ from $x = a$ to $x = b$ can be approximated by the area of trapezoid defined by the points $\{a, 0\}$, $\{a, f(a)\}$, $\{b, 0\}$, and $\{b, f(b)\}$ (Fig. 1.a). The corresponding formula is

$$\int_a^b f(x)\, dx \quad = \quad (b-a)\left[\frac{f(a)+f(b)}{2}\right] \,. \tag{3}$$

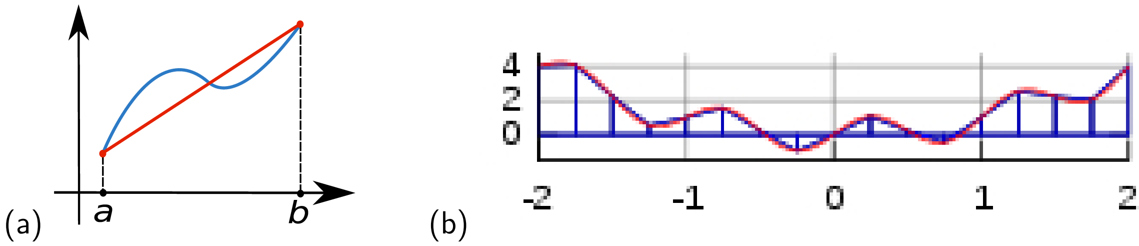

Figure 1: Illustration of the trapezoidal rule

The approximation is improved if the interval $[a, b]$ is divided into N equally sized subintervals, the integral in each of which is approximated by the trapezoid rule (Fig. 1.b)

$$\begin{aligned} \int_a^b f(x)\, dx &= \frac{b-a}{N} \sum_{k=1}^{N} \frac{f(x_k) + f(x_{k+1})}{2} \\ &= \frac{b-a}{2N} \left(f(x_1) + 2f(x_2) + 2f(x_3) + ... + 2f(x_N) + f(x_{N+1}) \right) . \end{aligned} \tag{4}$$

(a) Plot $f(x)$ for $x \in [0, 10]$ (1 P)

(b) Evaluate

$$\int_0^{10} f(x)\, dx$$

analytically. (Report the intermediate steps of the calculation.) Check your result using the integration command of MatLab. (1 P)

(c) Implement a program which integrates $f(x)$ in the interval $x \in [0, 10]$ using the trapezoid rule and N subintervals. (4 P)

(d) Plot the results of the numerical integration as a function of the number of subintervals N. How many subintervals are needed to get an estimate which diverges less than 1% form the analytical value? (2 P)